PAPER AAS 91-430

Periodic Gravitational Perturbations for Conversion Between Osculating and Mean Orbit Elements

Joseph R. Guinn

Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California

AAS/AIAA Astrodynamics Specialist Conference

Durango, Colorado August 19-22, 1991
AAS Publications Office, P.O. Box 28130, San Diego CA 92128

AAS 91-430

Periodic Gravitational Perturbations for Conversion Between Osculating and Mean Orbit Elements

Joseph R. Guinn†

Algorithms for converting between osculating and mean orbit elements are currently limited to computing the contribution due to the second zonal harmonic (J_2). This paper presents an improved conversion algorithm that includes the effects of all zonal, sectorial and tesseral harmonics, second order J_2, and third body gravitational perturbations. Mean elements are useful for preliminary orbit and maneuver design; however, for more precise work, such as groundtrack targeting, osculating elements are required. This improved conversion algorithm was developed to meet accuracy requirements for the TOPEX/Poseidon mission; but, additional use can be considered for satellites orbiting planets like Venus that do not have a dominant J_2. Results are presented from tests performed using the new algorithm with the planned TOPEX/Poseidon Earth orbit as well as the Mars Observer and proposed circular Magellan (Venus) orbits.

INTRODUCTION

Mean orbit elements are advantageous for trajectory design and maneuver planning since they can be propagated very quickly. Unfortunately, since the periodic behavior has been removed, mean elements do not describe the exact orbit at any given time. Using osculating elements gives an exact description of the orbit always but computation costs are significantly increased due to the numerical integration procedure required for propagation. To exploit the advantages of each, an accurate conversion between the two is required.

Existing algorithms for converting between osculating and mean orbit elements are

†Member of Technical Staff and Member AIAA at the Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive Pasadena, California 91109, (818) 354-0425

limited to computing the contribution of central body flattening. More precisely, accounting for the second zonal harmonic (J_2) of a spherical harmonic expansion for the shape of the central body. Several algorithms can be found in the literature for performing this task[1-5]. In this paper, expressions are presented for computing all first order, periodic central body aspherical and third body gravitational perturbations as well as the second order aspherical J_2 term (J_2^2). Also, the expressions are used to compute mean elements from osculating Earth, Venus and Mars satellite trajectories and the results are plotted and discussed.

For oblate planets like the Earth and Mars the J_2 aspherical perturbations dominate all other periodic effects on orbiting satellites. But, for nearly spherical planets like Venus the J_2 perturbations are of the same magnitude as some additional aspherical (zonal, sectorial and tesseral) terms.

The J_2 only conversions account for "short period" perturbations. In this paper, the term "short period" refers to perturbations with frequencies greater than once per satellite orbital revolution. When converting from osculating to mean orbit elements all periodic terms must be computed to obtain strictly mean elements. While the dominant perturbations in the satellite semi-major axis are due to the short period J_2 terms, longer period perturbations from the sectorial and tesseral harmonics yield significant effects in other elements. The term "medium period" is used to describe the perturbations with frequencies less than short period but greater than one revolution of the central body. Even longer periodic terms exist and can be computed with the expressions presented in this paper; however, only short and medium period aspherical terms are used for the test results presented.

Periodic perturbations are also caused by third bodies. Of course, the mass and relative geometry of the bodies determines the magnitude; but, for low Earth orbiters the sun and moon perturbations in semi-major axis are about the same size as the J_2^2 effects. Short period terms effect the satellite semi-major axis and longer period terms commensurate with the orbital periods of the third bodies contribute to variations in the satellite inclination. The longer period perturbations do not conform to the previous definition of medium period terms so they will simply be called "low frequency third body perturbations" in this paper.

APPLICATIONS

One motivations for this work is to determine a method for converting between osculating and mean orbit elements for the TOPEX/Poseidon Navigation Team. This team will perform preliminary maneuver design by propagating a set of mean elements over intervals of 100-200 days[6]. The mean semi-major axis, eccentricity, and inclination are required to be known to 1 meter, 10^{-5} and 0.001 degrees respectively. A frozen orbit (i.e., near zero mean rate in eccentricity and argument of periapsis) has been chosen for TOPEX/Poseidon[7]. From this requirement the mean argument of periapsis also must be known to ±10 degrees.

The expressions presented in this paper are derived from previous work related to averaged orbit element propagation[8-12]. Modifications to allow for normalized gravitational field coefficients and a simple summation of terms at a given time yields the periodic perturbations required in the conversion between osculating and mean orbit elements. Currently, the expressions are best suited for near circular orbits (i.e., $e < 0.1$) and no attempt has been made to reformulate them to eliminate singularities.

MATHEMATICAL DESCRIPTION

Expressions for computing the first order periodic aspherical and third body gravitational perturbations at a given time are presented in this section. Symbolically, the relationship between osculating and mean elements can be written as:

$$\alpha(t)_{mean} = \alpha(t)_{osculating} - \Delta\alpha(t) - \Delta\alpha_*(t)$$

where: α represents any orbit element at a particular epoch (t)
$\Delta\alpha$ sum of all periodic aspherical central body perturbations at (t)
$\Delta\alpha_*$ sum of all periodic third body perturbations at (t)

In the aspherical and third body perturbation expressions, classical orbit elements of a satellite orbit are used. They are defined as:

a = semi-major axis
e = eccentricity
i = inclination to central body equatorial plane
Ω = Longitude of ascending node
ω = argument of periapsis
M = mean anomaly

Additionally, the mean motion and gravitational parameter are defined as:

n = mean motion

μ = central body gravitational parameter
$= n^2 a^3$

In the expressions for the third body perturbations a superscript asterisk (e.g., $\mu*$) is used to distinguish third body variables.

Aspherical Gravity

Two formulations are combined to compute the periodic aspherical gravitational terms. The first method uses a time transformation and semi-equinoctial elements[10,11] to account for the J_2 and J_2^2 effects. This formulation is valid for all eccentric orbits and is free from singularities. The second method uses a spherical harmonic expansion of the central body potential to compute the terms for any degree and order of gravity harmonic[9]. The total aspherical perturbation is determined by computing the sum of all periodic terms of the two methods at a given time.

The formulation of the J_2 and J_2^2 effects is fully documented in the reference above and will not be repeated here. For the higher degree zonal, sectorial and tesseral terms the complete details are given below.

Starting with the normalized central body potential function[9]:

$$\mathfrak{R} = \sum_{\ell=2}^{\ell} \frac{\mu\, a_e^{\ell}}{a^{\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e)\ S_{\ell mpq}(\omega, M, \Omega, \Theta) \tag{1}$$

where:

a_e = central body equatorial radius

$\overline{F}_{\ell mp}(i)$ = normalized inclination function

$$= \sqrt{\frac{(\ell-m)!}{(\ell+m)!}(2\ell+1)(2-\delta)} \sum_{t=0}^{\min(p,k)} \frac{(2\ell-2t)!}{t!(\ell-t)!(\ell-m-2t)!2^{2\ell-2t}} \sin^{\ell-m-2t} i$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \cos^s i \sum_{c}^{\min(\ell-m-2t+s,\,p-t)} \binom{\ell-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k} \tag{2}$$

and: k = integer part of $(\ell-m)/2$

δ = Kronecker delta ($\delta=1$ for $m=0$, $\delta=0$ for $m\neq0$)

$G_{\ell pq}$ = eccentricity function

$$G_{\ell p-2} = (\ell^2+16p^2-8\ell p-4\ell+18p+4)e^2/8$$

$$G_{\ell p-1} = (-\ell+4p+1)e/2$$

$$G_{\ell p0} = 1+(-3\ell^2-16p^2+16\ell p+\ell)e^2/4$$

$$G_{\ell p1} = (3\ell-4p+1)e/2$$

$$G_{\ell p2} = (9\ell^2+16p^2-24\ell p+14\ell-18p+4)e^2/8 \tag{3}$$

$$S_{\ell mpq} = \begin{bmatrix} C_{\ell m} \\ -S_{\ell m} \end{bmatrix}_{\ell-m\ \mathrm{odd}}^{\ell-m\ \mathrm{even}} \cos\psi + \begin{bmatrix} S_{\ell m} \\ C_{\ell m} \end{bmatrix}_{\ell-m\ \mathrm{odd}}^{\ell-m\ \mathrm{even}} \sin\psi \tag{4}$$

and:

$$\psi = (\ell-2p)\omega + (\ell-2p+q)M + m(\Omega-\Theta) \tag{5}$$

The variable Θ is the angle along central body true equator-of-date between the prime meridian and vernal equinox at epoch.

Next, the time rates of change of classical orbit elements are obtained from Lagrange's planetary equations:

$$\frac{da}{dt}=\frac{2}{na}\frac{\partial \mathfrak{R}}{\partial M}$$

$$\frac{de}{dt}=\frac{1-e^2}{na^2e}\frac{\partial \mathfrak{R}}{\partial M}-\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial \mathfrak{R}}{\partial \omega}$$

$$\frac{di}{dt}=\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \mathfrak{R}}{\partial \omega}-\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \mathfrak{R}}{\partial \Omega}$$

$$\frac{d\Omega}{dt}=\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \mathfrak{R}}{\partial i}$$

$$\frac{d\omega}{dt}=\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial \mathfrak{R}}{\partial e}-\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \mathfrak{R}}{\partial i}$$

$$\frac{dM}{dt}=n-\frac{1-e^2}{na^2e}\frac{\partial \mathfrak{R}}{\partial e}-\frac{2}{na}\frac{\partial \mathfrak{R}}{\partial a} \qquad (6)$$

These terms taken alone give the mean secular rates due to an aspherical central body. The values can be computed by evaluating equations (6) for combinations of ℓ,m,p,q that give zero coefficients to $\omega, M, (\Omega-\Theta)$ in equation (5).

Now, the partial derivatives of the disturbing functions with respect to the classical orbit elements are:

$$\frac{\partial \mathfrak{R}}{\partial a}=\sum_{\ell=2}^{\ell}-(\ell+1)\frac{\mu a_e^\ell}{a^{\ell+2}}\sum_{m=0}^{\ell}\sum_{p=0}^{\ell}\overline{F}_{\ell mp}(i)\sum_{q=-\infty}^{q=+\infty}G_{\ell pq}(e)\ S_{\ell mpq}(\omega,M,\Omega,\Theta)$$

$$\frac{\partial \mathfrak{R}}{\partial e}=\sum_{\ell=2}^{\ell}\frac{\mu a_e^\ell}{a^{\ell+1}}\sum_{m=0}^{\ell}\sum_{p=0}^{\ell}\overline{F}_{\ell mp}(i)\sum_{q=-\infty}^{q=+\infty}\frac{\partial G_{\ell pq}(e)}{\partial e}S_{\ell mpq}(\omega,M,\Omega,\Theta)$$

$$\frac{\partial \mathfrak{R}}{\partial i}=\sum_{\ell=2}^{\ell}\frac{\mu a_e^\ell}{a^{\ell+1}}\sum_{m=0}^{\ell}\sum_{p=0}^{\ell}\frac{\partial \overline{F}_{\ell mp}(i)}{\partial i}\sum_{q=-\infty}^{q=+\infty}G_{\ell pq}(e)\ S_{\ell mpq}(\omega,M,\Omega,\Theta)$$

$$\frac{\partial \mathfrak{R}}{\partial \Omega}=\sum_{\ell=2}^{\ell}\frac{\mu a_e^\ell}{a^{\ell+1}}\sum_{m=0}^{\ell}\sum_{p=0}^{\ell}\overline{F}_{\ell mp}(i)\sum_{q=-\infty}^{q=+\infty}G_{\ell pq}(e)\ \frac{\partial S_{\ell mpq}(\omega,M,\Omega,\Theta)}{\partial \Omega}$$

$$\frac{\partial \mathfrak{R}}{\partial \omega} = \sum_{\ell=2}^{\ell} \frac{\mu\, a_e^{\ell}}{a^{\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e)\ \frac{\partial S_{\ell mpq}(\omega, M, \Omega, \Theta)}{\partial \omega}$$

$$\frac{\partial \mathfrak{R}}{\partial M} = \sum_{\ell=2}^{\ell} \frac{\mu\, a_e^{\ell}}{a^{\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e)\ \frac{\partial S_{\ell mpq}(\omega, M, \Omega, \Theta)}{\partial M} \qquad (7)$$

where:

$$\frac{\partial \overline{F}_{\ell mp}(i)}{di} = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}(2\ell+1)(2-\delta)} \sum_{t=0}^{\min(p,k)} (\ell\text{-}m\text{-}2t) \frac{(2\ell-2t)!}{t!(\ell-t)!(\ell-m-2t)!2^{2\ell-2t}} \sin^{\ell-m-2t-1} i$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \cos^{s} i \sum_{c}^{\min(\ell-m-2t+s,\,p-t)} \binom{\ell-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k}$$

$$+ \sqrt{\frac{(\ell-m)!}{(\ell+m)!}(2\ell+1)(2-\delta)} \sum_{t=0}^{\min(p,k)} \frac{(2\ell-2t)!}{t!(\ell-t)!(\ell-m-2t)!2^{2\ell-2t}} \sin^{\ell-m-2t} i$$

$$\times \sum_{s=0}^{m} \binom{m}{s} (-s) \cos^{s-1} i\ \ \sin i \sum_{c}^{\min(\ell-m-2t+s,\,p-t)} \binom{\ell-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k} \qquad (8)$$

$$\frac{\partial G_{\ell p\text{-}2}}{\partial e} = (\ell^2 + 16p^2 - 8\ell p - 4\ell + 18p + 4)e/4$$

$$\frac{\partial G_{\ell p\text{-}1}}{\partial e} = (-\ell + 4p + 1)/2$$

$$\frac{\partial G_{\ell p0}}{\partial e} = 1 + (-3\ell^2 - 16p^2 + 16\ell p + \ell)e/2$$

$$\frac{\partial G_{\ell p1}}{\partial e} = (3\ell - 4p + 1)/2$$

$$\frac{\partial G_{\ell p2}}{\partial e} = (9\ell^2 + 16p^2 - 24\ell p + 14\ell - 18p + 4)e/4 \qquad (9)$$

By letting:

$$\widetilde{S}_{\ell mpq} = \int S_{\ell mpq} d\psi = \begin{bmatrix} C_{\ell m} \\ -S_{\ell m} \end{bmatrix}_{\ell-m\text{ odd}}^{\ell-m\text{ even}} \sin\psi - \begin{bmatrix} S_{\ell m} \\ C_{\ell m} \end{bmatrix}_{\ell-m\text{ odd}}^{\ell-m\text{ even}} \cos\psi \tag{10}$$

then:

$$\frac{\partial S_{\ell mpq}}{\partial \Omega} = -m\widetilde{S}_{\ell mpq} \tag{11}$$

$$\frac{\partial S_{\ell mpq}}{\partial \omega} = -(\ell-2p)\widetilde{S}_{\ell mpq} \tag{12}$$

$$\frac{\partial S_{\ell mpq}}{\partial M} = -(\ell-2p+q)\widetilde{S}_{\ell mpq} \tag{13}$$

Taking the semi-major axis as an example, the periodic perturbations can be computed by integrating over one cycle of ψ.

$$\Delta a = \int \frac{da}{dt} dt = \int \frac{da}{dt}\frac{d\psi}{\dot{\psi}} = \frac{1}{\dot{\psi}}\int \frac{2}{na}\frac{\partial \Re}{\partial M} d\psi \tag{14}$$

with:

$$\dot{\psi} = (\ell-2p)\dot{\omega} + (\ell-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\Theta}) \tag{15}$$

Substituting equations (6) into (14) yields the sum of all periodic perturbations in each orbit element.

$$\Delta a = \frac{2na}{\dot{\psi}} \sum_{\ell=2}^{\ell} \left(\frac{a_e}{a}\right)^{\ell} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp} \sum_{q=-\infty}^{q=+\infty} G_{\ell pq} (\ell-2p+q) S_{\ell mpq}$$

$$\Delta e = \frac{n\sqrt{1-e^2}}{\dot{\psi} e} \sum_{\ell=2}^{\ell} \left(\frac{a_e}{a}\right)^{\ell} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp} \sum_{q=-\infty}^{q=+\infty} G_{\ell pq} \left[\sqrt{1-e^2}(\ell-2p+q) - (\ell-2p)\right] S_{\ell mpq}$$

$$\Delta i = \frac{n}{\dot{\psi}\sqrt{1-e^2}\sin i} \sum_{\ell=2}^{\ell}\left(\frac{a_e}{a}\right)^{\ell}\sum_{m=0}^{\ell}\sum_{p=0}^{\ell}\overline{F}_{\ell mp}\sum_{q=-\infty}^{q=+\infty}G_{\ell pq}\left[(\ell-2p)\cos i - m\right]S_{\ell mpq}$$

$$\Delta\Omega = \frac{n}{\dot{\psi}\sqrt{1-e^2}\sin i} \sum_{\ell=2}^{\ell}\left(\frac{a_e}{a}\right)^{\ell}\sum_{m=0}^{\ell}\sum_{p=0}^{\ell}\frac{\partial\overline{F}_{\ell mp}}{\partial i}\sum_{q=-\infty}^{q=+\infty}G_{\ell pq}\,\widetilde{S}_{\ell mpq}$$

$$\Delta\omega = \frac{n\sqrt{1-e^2}}{\dot{\psi}} \sum_{\ell=2}^{\ell}\left(\frac{a_e}{a}\right)^{\ell}\sum_{m=0}^{\ell}\sum_{p=0}^{\ell}\sum_{q=-\infty}^{q=+\infty}\left[\frac{1}{e}\overline{F}_{\ell mp}\frac{\partial G_{\ell pq}}{\partial e} - \frac{\cot i}{1-e^2}\frac{\partial\overline{F}_{\ell mp}}{\partial i}G_{\ell pq}\right]\widetilde{S}_{\ell mpq}$$

$$\Delta M = \frac{-n}{\dot{\psi}} \sum_{\ell=2}^{\ell}\left(\frac{a_e}{a}\right)^{\ell}\sum_{m=0}^{\ell}\sum_{p=0}^{\ell}\overline{F}_{\ell mp}\sum_{q=-\infty}^{q=+\infty}\left[\frac{1-e^2}{e}\frac{\partial G_{\ell pq}}{\partial e} - 2(\ell+1)G_{\ell pq}\right.$$
$$\left. + \frac{3(\ell-2p+q)n}{\dot{\psi}}G_{\ell pq}\right]\widetilde{S}_{\ell mpq} \tag{16}$$

The expression for mean anomaly does not include the central body term and is augmented with a second order correction. The second order correction[9] is added to account for changes in the mean motion arising from perturbations in the semi-major axis.

Third Body Gravity

Analogous to the aspherical gravity the third body periodic gravitational terms are derived from a disturbing function. The disturbing function for this work was obtained from Kaula[8]. A minor modification was introduced to make use of the normalized inclination function. The new third body disturbing function is:

$$\mathfrak{R}_* = \sum_{\ell=2}^{\ell}\frac{\mu_*\,a^{\ell}}{(2\ell+1)a_*^{\ell+1}}\sum_{m=0}^{\ell}\sum_{p=0}^{\ell}\overline{F}_{\ell mp}(i)\sum_{q=-\infty}^{q=+\infty}G_{\ell pq}(e)\sum_{h=0}^{\ell}\overline{F}_{\ell mh}(i_*)\sum_{j=-\infty}^{j=+\infty}G_{\ell hj}(e_*)\cos\psi_* \tag{17}$$

with:

$$\psi_* = (\ell-2p)\omega + (\ell-2p+q)M - (\ell-2h)\omega_* - (\ell-2h+j)M_* + m(\Omega-\Omega_*) \tag{18}$$

Again like aspherical gravity, the partials of the disturbing function with respect to the orbit elements are computed[12]:

$$\frac{\partial \mathfrak{R}*}{\partial a} = \sum_{\ell=2}^{\ell} \frac{\mu* \ell a^{\ell-1}}{(2\ell+1)\, a*^{\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e) \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i*) \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e*)\ \cos\psi*$$

$$\frac{\partial \mathfrak{R}*}{\partial e} = \sum_{\ell=2}^{\ell} \frac{\mu* a^{\ell}}{(2\ell+1) a*^{\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} \frac{\partial G_{\ell pq}(e)}{\partial e} \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i*) \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e*)\ \cos\psi*$$

$$\frac{\partial \mathfrak{R}*}{\partial i} = \sum_{\ell=2}^{\ell} \frac{\mu* a^{\ell}}{(2\ell+1) a*^{\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \frac{\partial \overline{F}_{\ell mp}(i)}{\partial i} \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e) \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i*) \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e*)\ \cos\psi*$$

$$\frac{\partial \mathfrak{R}*}{\partial \Omega} = \sum_{\ell=2}^{\ell} \frac{\mu* a^{\ell}}{(2\ell+1) a*^{\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e) \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i*) \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e*) \left[-(m)\sin\psi*\right]$$

$$\frac{\partial \mathfrak{R}*}{\partial \omega} = \sum_{\ell=2}^{\ell} \frac{\mu* a^{\ell}}{(2\ell+1) a*^{\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e) \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i*) \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e*) \left[-(\ell-2p)\sin\psi*\right]$$

$$\frac{\partial \mathfrak{R}*}{\partial M} = \sum_{\ell=2}^{\ell} \frac{\mu* a^{\ell}}{(2\ell+1) a*^{\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e) \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i*) \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e*) \left[-(\ell-2p+q)\sin\psi*\right]$$

(19)

The time derivative of (18) is:

$$\dot{\psi}* = (\ell\text{-}2p)\dot{\omega} + (\ell\text{-}2p+q)\dot{M} - (\ell\text{-}2h)\dot{\omega}* - (\ell\text{-}2h+j)\dot{M}* + m(\dot{\Omega}-\dot{\Omega}*) \tag{20}$$

Substituting equations (19) into (6) and integrating, the periodic third body expressions are produced:

$$\Delta a* = \frac{2}{na^{*}\dot{\psi}*} \sum_{\ell=2}^{\ell} \frac{\mu* a^{\ell}}{(2\ell+1) a*^{\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e) \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i*)$$

$$\times \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e*)\ (\ell\text{-}2p+q)\cos\psi*$$

$$\Delta e^* = \frac{\sqrt{1-e^2}}{na^{*2} e^* \dot{\psi}^*} \sum_{\ell=2}^{\ell} \frac{\mu^* a^{\ell}}{(2\ell+1)a^{*\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e) \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i^*)$$

$$\times \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e^*) \left[\sqrt{1-e^2}(\ell-2p+q) - (\ell-2p)\right] \cos\psi^*$$

$$\Delta i^* = \frac{1}{\dot{\psi}^* na^{*2}\sqrt{1-e^2}\sin i} \sum_{\ell=2}^{\ell} \frac{\mu^* a^{\ell}}{(2\ell+1)a^{*\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e) \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i^*)$$

$$\times \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e^*) \left[\cos i\,(\ell-2p) - m\right] \cos\psi^*$$

$$\Delta\Omega^* = \frac{1}{\dot{\psi}^* na^{*2}\sqrt{1-e^2}\sin i} \sum_{\ell=2}^{\ell} \frac{\mu^* a^{\ell}}{(2\ell+1)a^{*\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \frac{\partial \overline{F}_{\ell mp}(i)}{\partial i} \sum_{q=-\infty}^{q=+\infty} G_{\ell pq}(e)$$

$$\times \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i^*) \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e^*) \sin\psi^*$$

$$\Delta\omega^* = \frac{\sqrt{1-e^2}}{\dot{\psi}^* na^{*2}} \sum_{\ell=2}^{\ell} \frac{\mu^* a^{\ell}}{(2\ell+1)a^{*\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q=-\infty}^{q=+\infty} \left[\frac{1}{e} \overline{F}_{\ell mp}(i) \frac{\partial G_{\ell pq}(e)}{\partial e}\right.$$

$$\left. - \frac{\cot i}{1-e^2} \frac{\partial \overline{F}_{\ell mp}(i)}{\partial i} G_{\ell pq}(e)\right] \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i^*) \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e^*) \sin\psi^*$$

$$\Delta M^* = \frac{-1}{\dot{\psi}^* na^{*2}} \sum_{\ell=2}^{\ell} \frac{\mu^* a^{\ell}}{(2\ell+1)a^{*\ell+1}} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \overline{F}_{\ell mp}(i) \sum_{q=-\infty}^{q=+\infty} \left[\frac{1-e^2}{e} \frac{\partial G_{\ell pq}(e)}{\partial e}\right.$$

$$\left. + 2\ell G_{\ell pq}(e)\right] \sum_{h=0}^{\ell} \overline{F}_{\ell mh}(i^*) \sum_{j=-\infty}^{j=+\infty} G_{\ell hj}(e^*) \sin\psi^* \qquad (21)$$

TEST RESULTS

Tests of the above expressions were performed by starting with osculating trajectories that only included central and third body perturbations. No atmospheric drag or solar radiation pressure forces were introduced. The osculating trajectories were generated with the initial conditions given in Table 1. The gravity field size and third body perturbations are shown in Table 2.

Table 1
INITIAL CONDITIONS
(Epoch = 22 June 1992)

	TOPEX/POSEIDON	MAGELLAN	MARS OBSERVER
a (km)	7720.3855	6376.0000	3766.1588
e	3.43×10^{-4}	3.07×10^{-2}	4.0×10^{-3}
i (deg)	66.049	85.500	92.869
Ω (deg)	116.5500	0.0000	38.3372
ω (deg)	329.5517	90.0000	270.0000
M (deg)	13.5615	0.00000	0.0000

Table 2
PERTURBATION MODELS

	TOPEX/POSEIDON	MAGELLAN	MARS OBSERVER
Central Body Gravity Field Size	17x17	10,10	18,18
Third Body	Sun,Moon	Sun	Sun

Time series of the osculating and converted mean elements were generated and are shown in Figs. 1-9. In the plots, the legend "Mean(J2)" implies that only the J_2 perturbations have been subtracted from the osculating element. Likewise the abbreviations "CS" (all aspherical central body terms except J_2), "J2**2", (second order J_2), and "3B" (third body terms) are used.

Semi-Major Axis *(a)*

Figs. 1-3 show the progressive removal of periodic perturbations from the osculating semi-major axis. Only short period perturbations (i.e., terms with ℓ-2p+q≠0) are removed. For TOPEX/Poseidon, Figs. 1a-1d, removing aspherical and third body perturbations yields a mean semi-major axis with an uncertainty of ±1 meter. In Figs. 2a-2d (Venus orbiter) the J_2 only conversion gives a mean value that is as uncertain as the osculating. This verifies the importance of the conversion with aspherical terms beyond J_2 for nearly spherical planets. For Magellan orbit tested a mean semi-major axis of ±15 meters is possible when the complete first order aspherical gravitational effects are included. For Mars Observer the conversions do not perform as well. In Figs. 3a-3d the mean value of the semi-major axis has an uncertainty of ±50-60 meters. The results are worse due to the first order approximation in aspherical gravity field coefficients beyond J_2. That is, the Mars J_2^2, C_{22}^2, S_{22}^2, etc. have been neglected. Fig. 3c indicates that the perturbations due to J_2^2 do not dominant among the neglected terms and thus a more complete expansion is required to resolve the uncertainty.

For all three satellites the sun perturbations are smaller than the uncertainty produced by the aspherical conversions. But, for TOPEX/Poseidon the Moon perturbations contribute about 50 centimeters.

Eccentricity *(e)*

The mean eccentricities derived from the aspherical gravitational terms are shown in Figs. 4a-4f. The third body perturbations in eccentricity are also shown but cannot be distinguished since they produce perturbations of less than 10^{-6} for all three satellites. In Figs. 4b,4d and 4f medium period perturbations have been removed. These are computed from terms with ℓ-2p+q=0 and $m \neq 0$. The large difference in Fig. 4d is due to a medium period aspherical gravity term caused by the slow rotation of Venus. The periodic nature is not obvious since medium period frequencies for Venus can be as long as 243 Earth days.

Inclination *(i)*

The conversion from osculating to mean inclination involves computing short period aspherical terms as well as perturbations commensurate with the rotation period of the central body and the orbital motion of any third bodies. Figs. 5a-5f primarily show the effects of the short period aspherical terms. The medium period aspherical effects are also easily seen in Figs. 5b, 5e and 5f. The slow rotation of Venus is again the reason the periodic terms in Fig. 5d are not seen.

Low frequency third body perturbations can be seen in Fig. 5b; but, more dramatic views are shown in Figs 6a-6f. In particular, Fig. 6b shows the low frequency lunar and solar terms at about 11 and 60 days respectively. The medium period aspherical gravity terms dominate for Venus and Mars thus masking, in Figs. 6d and 6f, the third body perturbations.

Longitude of Ascending Node *(Ω)*

Due to the secular nature of the longitude of the ascending node it is difficult to examine graphically the difference between the mean and osculating values. Figs. 7a-7f show only the conversions due to aspherical gravity since the third body perturbations are small. The plots are limited to a single orbit and the differences are included to remove the secular effects.

Aspherical gravity induces a torque on satellite orbits that in turn gives rise to a precession of Ω. This precession is the basis for the secular rate. For Venus, the secular rate is very small as can be seen in Figs. 7c and 7d. After removing the aspherical gravity perturbations the mean value of Ω is determined with a constant mean secular rate.

Argument of Periapsis *(ω)*

Again this element has secular variations. Similar to eccentricity, the argument of periapsis has only medium period variations due to aspherical effects past J_2. It is also practically insensitive to third body perturbations. Figs. 8a-8f show the osculating and mean values over one day.

Mean Anomaly *(M)*

This element changes very rapidly due to the central body term. The short period variations are primarily dominated by the J_2 perturbations. Figs. 9a-9f show one orbital revolution of the osculating and mean values with their differences.

CONCLUSIONS

An improved algorithm for converting between osculating and mean orbit elements has been developed and the expressions provided in this paper. A summary of the uncertainties in the mean elements determined with the new expressions is given in Table 3.

The primary goal of determining the TOPEX/Poseidon mean semi-major axis, eccentricity, and inclination to 1 meter, 10^{-5} and 0.001 degrees respectively has been achieved. The additional requirement on the mean argument of periapsis, ±10 degrees, was also meet.

Table 3
MEAN ELEMENTS AND UNCERTAINTIES
(Epoch = 22 June 1992)

	TOPEX/POSEIDON		MAGELLAN		MARS OBSERVER	
a (km)	7714.4278	(0.001)	6376.020	(0.015)	3774.807	(0.050)
e	9.5×10^{-5}	(2×10^{-5})	2.473×10^{-2}	(2×10^{-5})	7.14×10^{-3}	(3×10^{-4})
i (deg)	66.039	(0.002)	85.302	(0.002)	92.898	(0.002)
Ω *(deg)*	116.5574	(10^{-5})	0.0003	(10^{-5})	38.3372	(10^{-3})
ω *(deg)*	81.4	(9)	98.8	(0.2)	270.6	(4)
$\bar{M}$ *(deg)*	253.1300	(10^{-3})	0.0004	(2×10^{-4})	0.0000	(10^{-3})

Fig.1
SEMI-MAJOR AXIS
TOPEX/POSEIDON (EARTH)

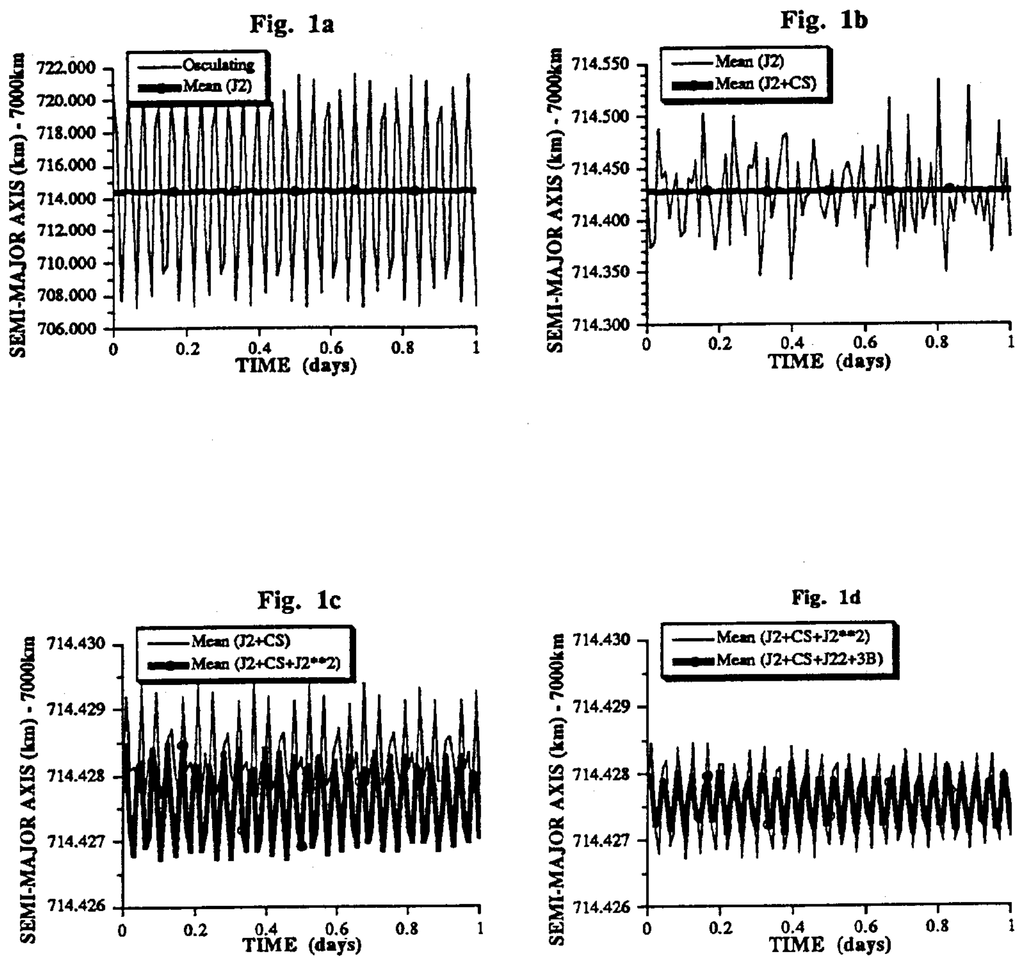


Fig.2
SEMI-MAJOR AXIS
MAGELLAN (VENUS)

Fig. 2a

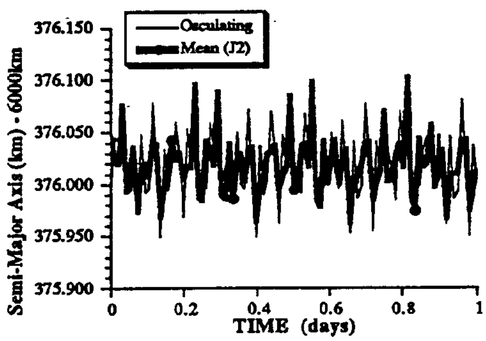


Fig. 2b

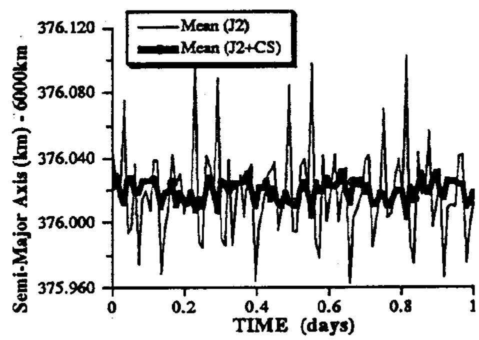


Fig. 2c

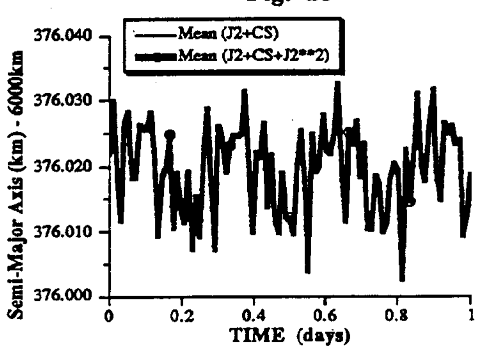


Fig. 2d

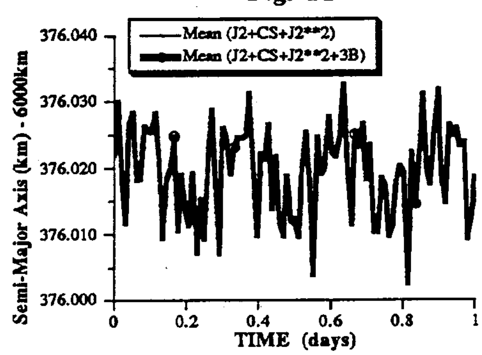


Fig.3
SEMI-MAJOR AXIS
MARS OBSERVER

Fig. 3a

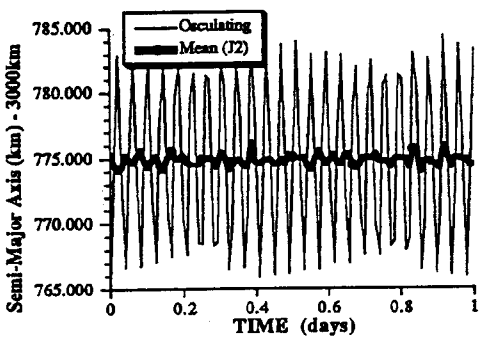


Fig. 3b

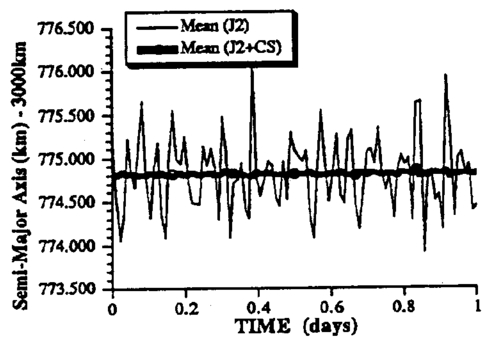


Fig. 3c

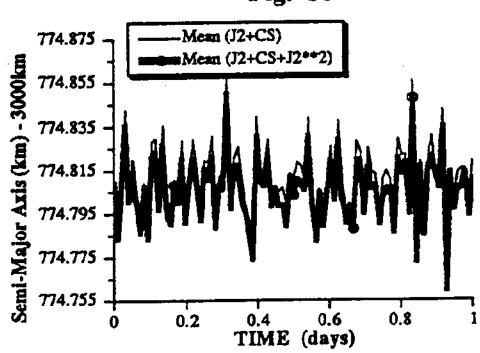


Fig. 3d

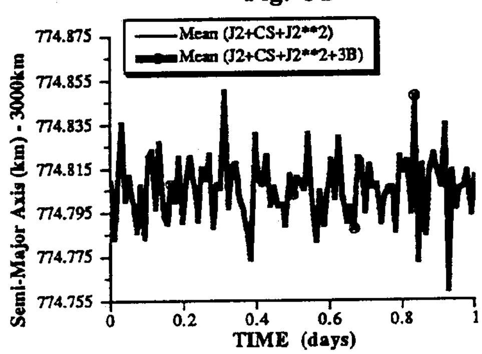


Fig.4
ECCENTRICITY

Fig. 4a

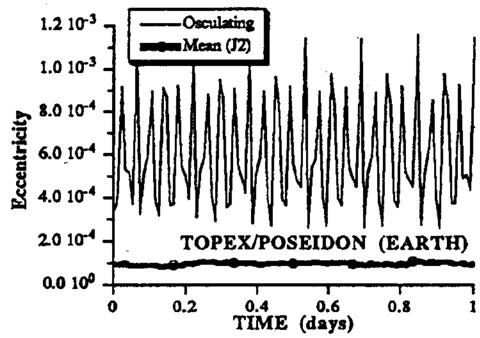


Fig. 4b

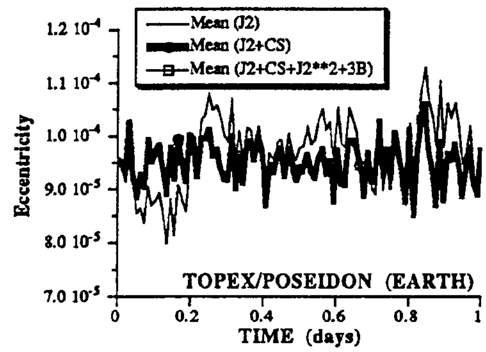


Fig. 4c

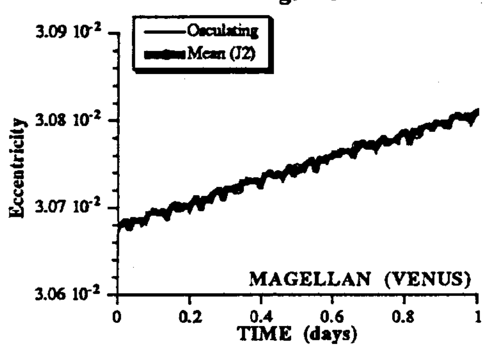


Fig. 4d

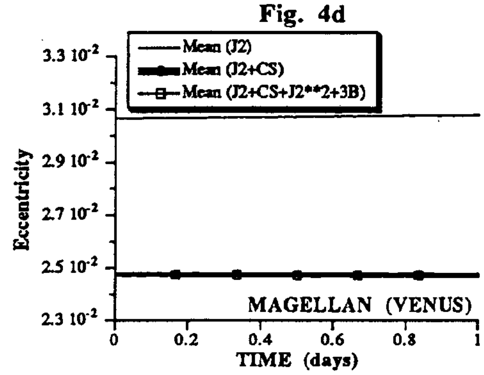


Fig. 4e

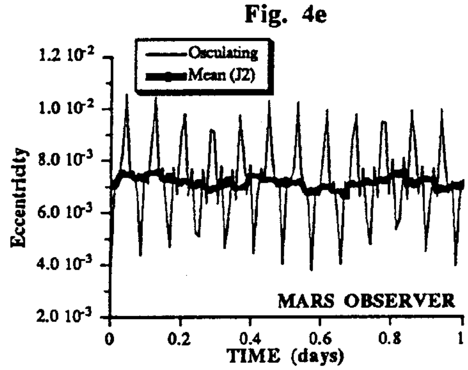


Fig. 4f

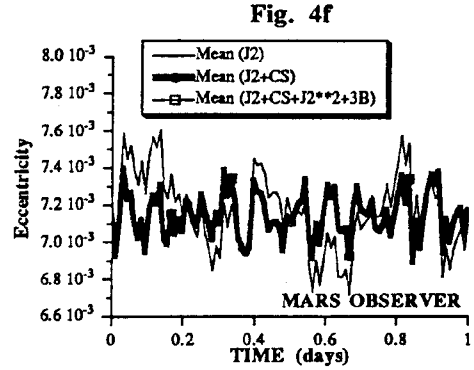


Fig.5
INCLINATION

Fig. 5a

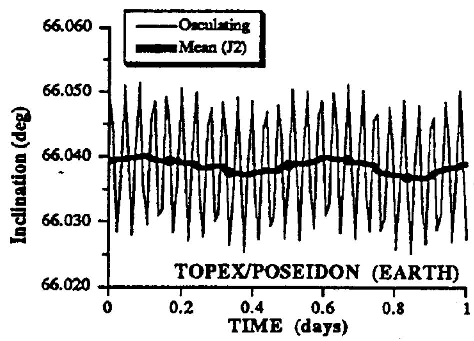


Fig. 5b

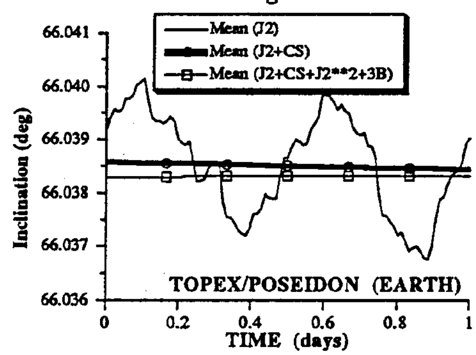


Fig. 5c

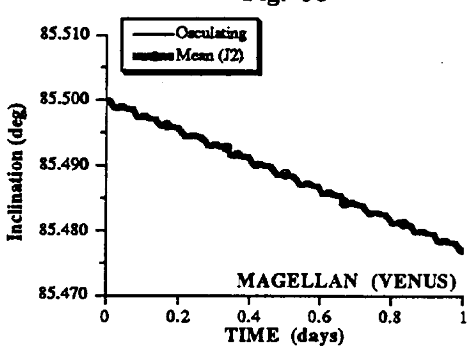


Fig. 5d

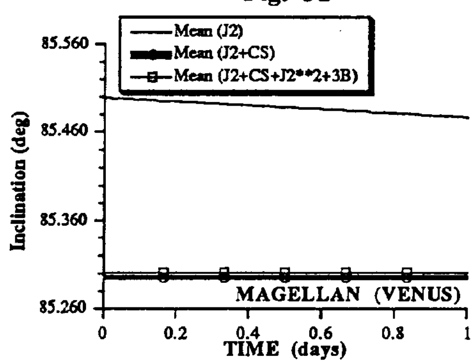


Fig. 5e

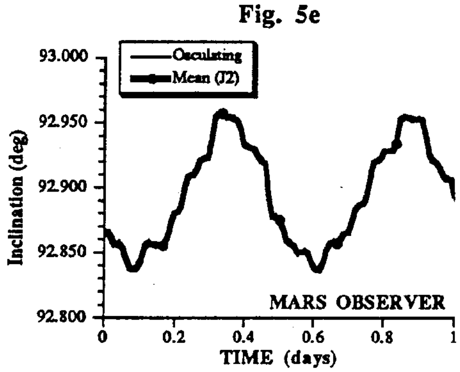


Fig. 5f

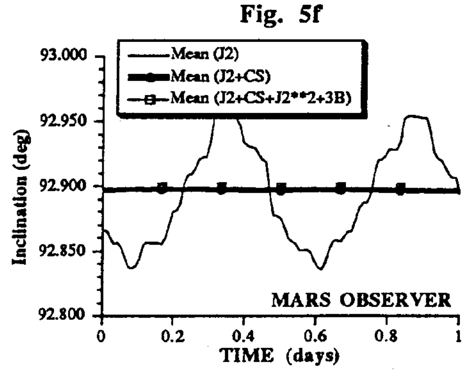


Fig.6
INCLINATION

Fig. 6a

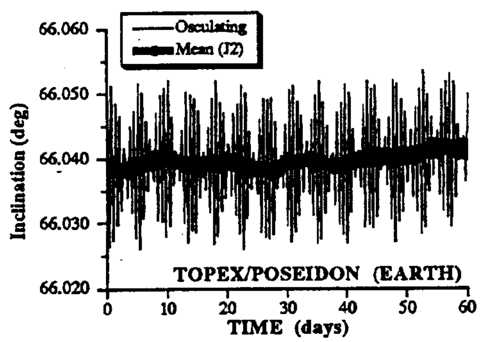


Fig. 6b

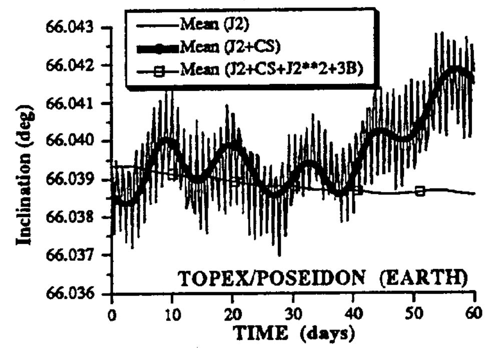


Fig. 6c

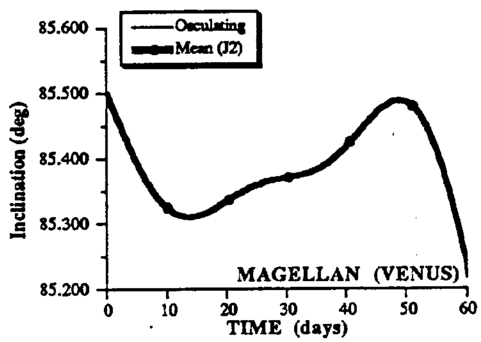


Fig. 6d

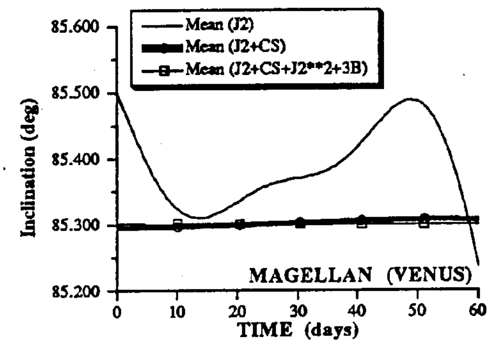


Fig. 6e

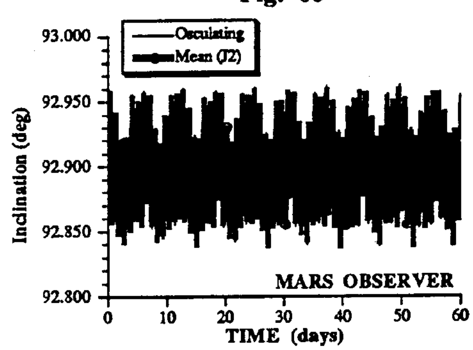


Fig. 6f

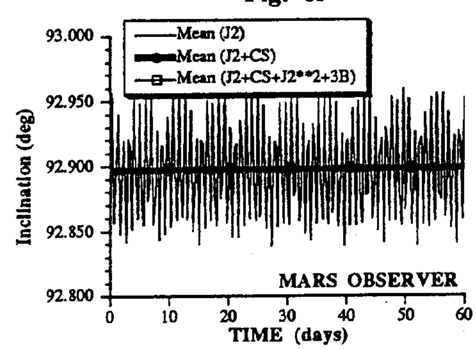


Fig.7
LONGITUDE OF ASCENDING NODE

Fig. 7a

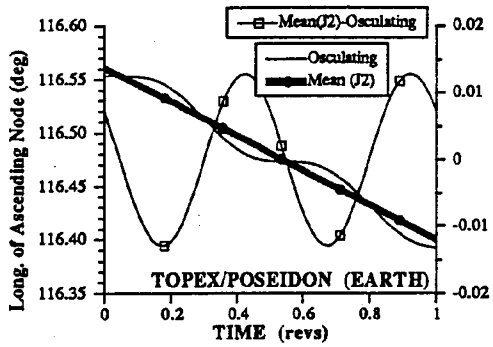


Fig. 7b

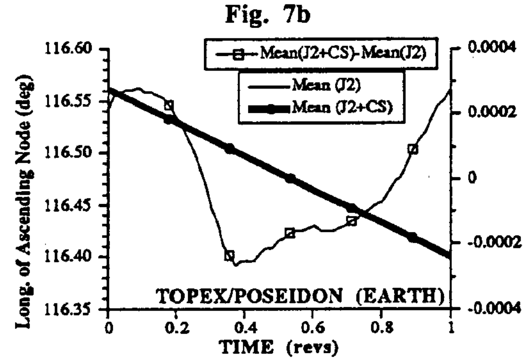


Fig. 7c

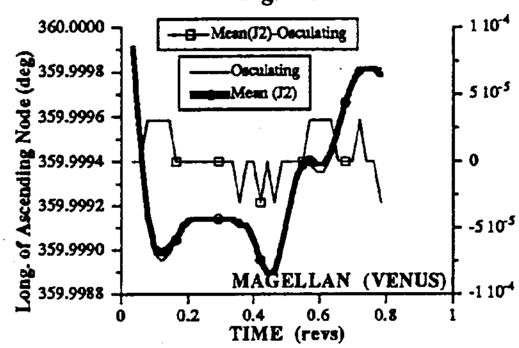


Fig. 7d

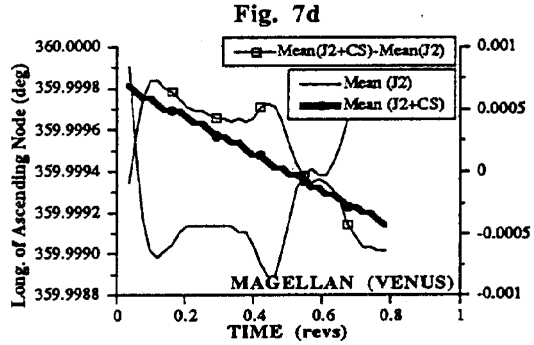


Fig. 7e

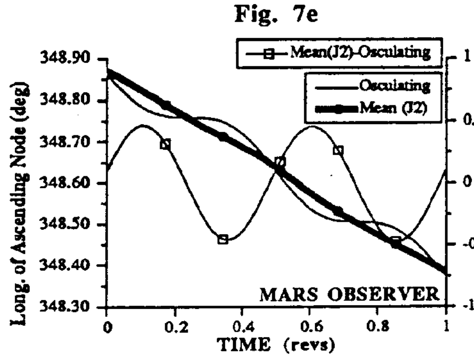


Fig. 7f

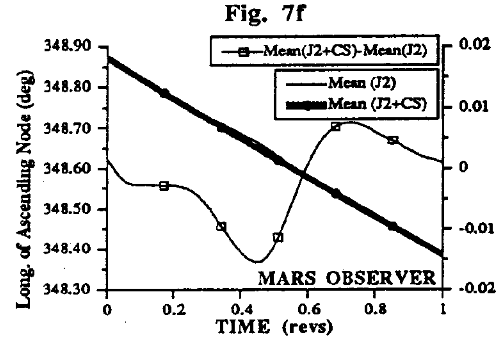


Fig.8
ARGUMENT OF PERIAPSIS

Fig. 8a

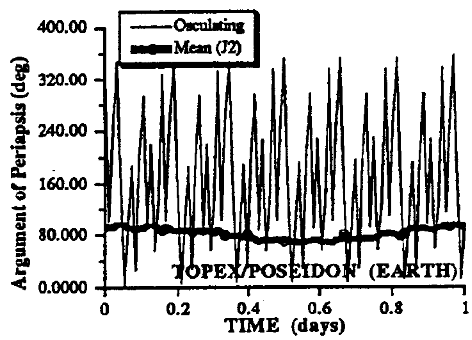


Fig. 8b

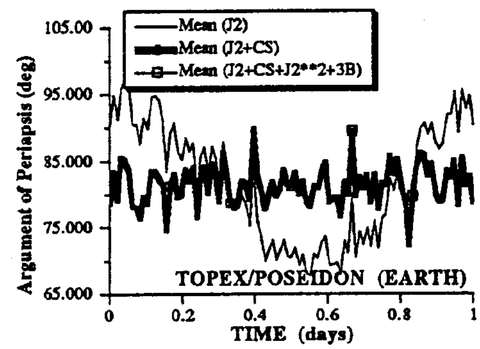


Fig. 8c

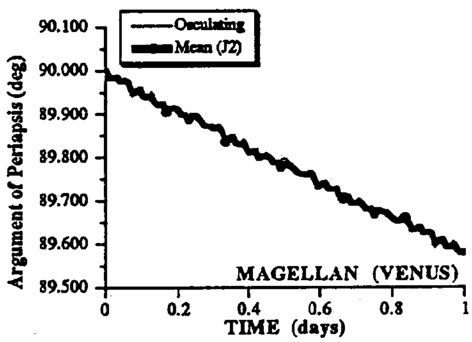


Fig. 8d

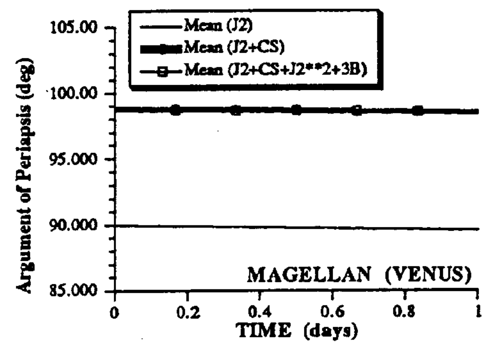


Fig. 8e

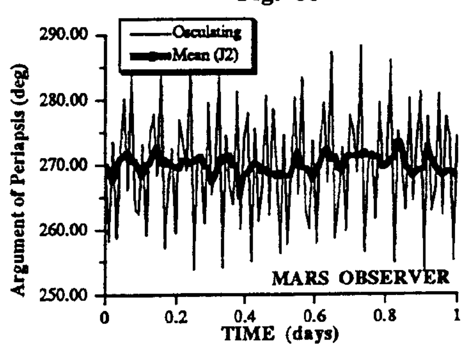


Fig. 8f

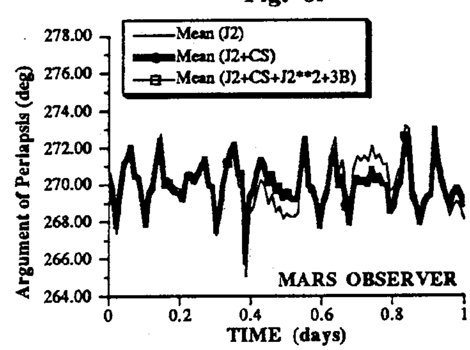


Fig.9
MEAN ANOMALY

Fig. 9a

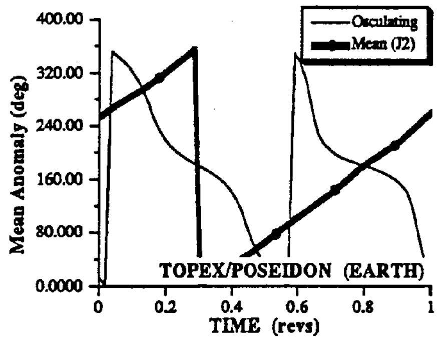


Fig. 9b

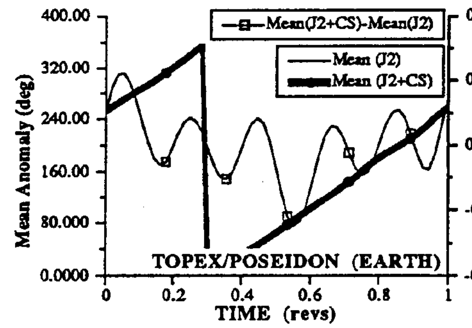


Fig. 9c

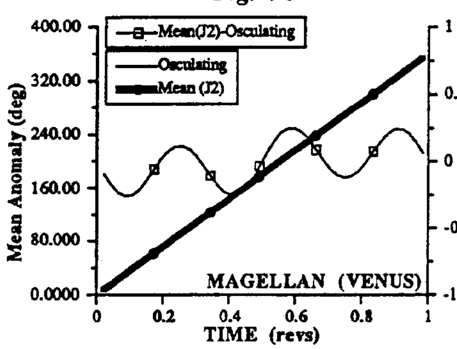


Fig. 9d

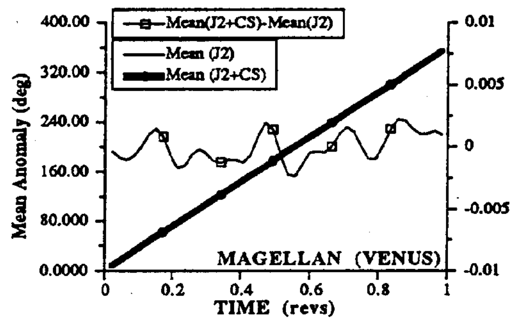


Fig. 9e

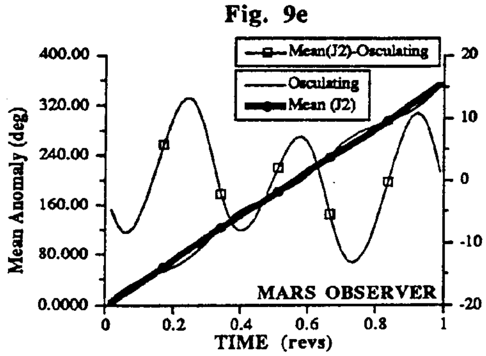
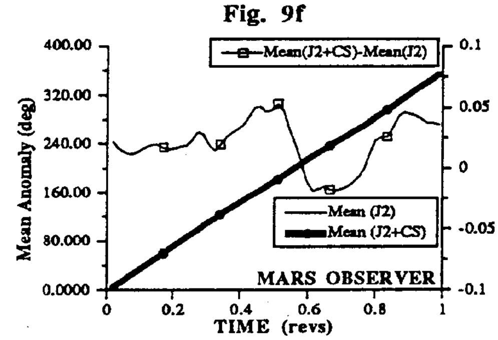


Fig. 9f



ACKNOWLEDGEMENTS

The research described in this paper was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

REFERENCES

1) Brouwer, D., "Solution of the Problem of Artificial Satellite Theory Without Drag," *The Astronomical Journal*, **64**, No.1274, pp. 378-397, (1959).
2) Kozai, Y., "The Motion of a Close Earth Satellite," *The Astronomical Journal*, **64**, No.1274, pp. 367-377, (1959).
3) Izsak, I.G., "A Note on Perturbation Theory", The Astronomical Journal, 68, No.8, pp. 559-561, (1963).
4) Cain, B.J., "Determination of Mean Elements for Brouwer's Satellite Theory," *The Astronomical Journal*, **67**, No.6, pp. 391-392, (1967).
5) Walter, H.G., "Conversion of Osculating Orbital Elements into Mean Elements," *The Astronomical Journal*, **72**, No.8, pp. 994-997, (1967).
6) Bhat,R.S.,R.B.Frauenholz, "TOPEX/POSEIDON Orbit Maintenance Maneuver Design," *Presented at the AAS/AIAA Astrodynamics Specialist Conference,* AAS 89-408, Stowe, Vermont, 7-10 August 1989.
7) Vincent, M.A., "Reference Orbit #5," JPL IOM 312/89.6-052 (Internal Document), 22 November 1989.
8) Kaula, W.M., "Development of the Lunar and Solar Disturbing Functions for a Close Satellite," *The Astronomical Journal*, **67**, No.5, pp. 300-303, (1962)
9) Kaula, W.M., "Theory of Satellite Geodesy," Blaisdell, Waltham, MA., 1966
10) Konopliv, A., "A Perturbation Method and Some Applications," *Celestial Mechanics*, **47**, pp. 305-320, (1990).
11) Konopliv, A., "A Third Order J2 Solution with a Transformed Time," JPL IOM 314.3-970 (Internal Document), 28 March 1991.
12) Bhat, R., Personal Communication, 1990.